# DISCONTINUOUS SOLUTIONS OF SPACE PROBLEMS IN THE THEORY OF IDEAL PLASTICITY

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This paper deals with discontinuous solutions of three-dimensional problems in the theory of ideal plasticity, for the case when the plastic state of stress corresponds to the edge of the prism, which interprets the Tresca-Saint Venant's condition of plasticity in the space of principal stresses.

It is noted that the discontinuous solutions in the theory of ideal plasticity were originally brought to attention by Khristianovich [1]. More recent and detailed investigations of this problem are credited to Sokolovskii [2], Prager [3,4], Winzer and Carrier [5], Lee [6], Hodge [7], Hill [8], Shapiro [9], and others.

1. Let us consider surfaces of discontinuity in axisymmetrical problems in the theory of ideal plasticity under the condition of complete plasticity. Let us direct the $z$-axis along the axis of rotation and let the $\rho$-axis be perpendicular to it. Obviously, it will be sufficient to investigate the distribution of stresses in the $\rho z$ plane.

In this case, the Tresca-Saint Venant condition of plasticity is of the form

$$(\sigma_\rho - \sigma_z)^2 + 4\tau_{\rho z}^2 = 4k^2 \qquad \sigma_\theta = \tfrac{1}{2}(\sigma_\rho + \sigma_z) + k \tag{1.1}$$

and, therefore, the following relationships will exist:

$$\begin{aligned} \sigma_\rho &= 2k\omega + k\sin 2\theta, \qquad & \tau_{\rho z} &= -k\cos 2\theta \\ \sigma_z &= 2k\omega - k\sin 2\theta, \qquad & \sigma_\theta &= 2k\omega + k \end{aligned} \tag{1.2}$$

Relationships (1.1) and (1.2) are completely analogous to the corresponding relationships for the plane problem. This analogy permits the extension of the basic relationships between stresses at the lines of discontinuity, which were obtained by Prager [3] for the plane problem, to the case of axisymmetrical problem considered here.

However, because of the different character of the equilibrium equations, solutions of problems for the case of plane deformation may not be directly extended to the case of axisymmetrical problems. For the case of plane problem, equations of equilibrium represent differential relationships which could be satisfied with any choice of constants $\omega$ and $\theta$. The equations of equilibrium for axisymmetrical problems

$$\frac{\partial \sigma_\rho}{\partial \rho}+\frac{\partial \tau_{\rho z}}{\partial z}+\frac{\sigma_\rho-\sigma_\theta}{\rho}=0, \qquad \frac{\partial \tau_{\rho z}}{\partial \rho}+\frac{\partial \sigma_z}{\partial z}+\frac{\tau_{\rho z}}{\rho}=0 \qquad (1.3)$$

This makes impossible the direct extension of such results of a plane problem as a discontinuous solution for a wedge, or any consequences of such a solution.

Therefore, the problem of finding discontinuous solutions for axisymmetrical problems leads to the complicated problem of matching regions with variable stresses.

Note that equations (1.3) are satisfied for a particular case when

$$\omega = \text{const}, \qquad \theta = \frac{\pi}{4} \pm \pi m \qquad (m = 0, 1, \ldots)$$

2. Assume that a certain surface $S$ is a discontinuity surface of a stressed state. This surface $S$ may separate states of stress corresponding to different edges of the prism which interprets the Tresca-Saint Venant condition of plasticity in the space of principal stresses. Further in this paper this prism will be called the Tresca-Saint Venant prism.

Obviously, it will be sufficient to investigate the cases when on either side of the surface $S$ one of the conditions

$$\sigma_1 = \sigma_2 = \sigma_3 \pm 2k \qquad (2.1)$$

is satisfied, inasmuch as this may always be attained by suitable labelling of the principal stresses. Two basic cases will be considered below. First, when the discontinuity surface $S$ separates plastic states of stress for which, in (2.1), the signs of the constant $2k$ are the same on either side of $S$. Second, the case when these signs are opposite.

Let us investigate the first case. At a certain point of the surface let us define an orthogonal system of coordinates $\alpha$, $\beta$, $n$. The $n$-axis will be directed along the normal to the surface $S$. Then the axes $\alpha$ and $\beta$ will be located in the tangent plane to the surface $S$.

Consider at the same point the directions of principal stresses $\sigma_1$, $\sigma_2$, $\sigma_3$.

The mutual orientation of axes $\alpha$, $\beta$, $n$ and axes 1, 2, 3, which define the directions of principal stresses, will be specified by the direction cosines, which are presented in the table:

| | 1 | 2 | 3 |
|---|---|---|---|
| $\alpha$ | $l_1$ | $m_1$ | $n_1$ |
| $\beta$ | $l_2$ | $m_2$ | $n_2$ |
| $n$ | $l_3$ | $m_3$ | $n_3$ |

Further, having related all the stress components to the constant $\pm\ 2k$, we obtain

$$\sigma_\alpha = \sigma_1 + n_1^2, \qquad \sigma_\beta = \sigma_1 + n_2^2, \qquad \sigma_n = \sigma_1 + n_3^2$$

$$\tau_{\alpha\beta} = n_1 n_2, \qquad \tau_{\alpha n} = n_1 n_3, \qquad \tau_{\beta n} = n_2 n_3$$

where

$$\sigma_1 = \sigma - {}^1/_3, \quad \sigma = {}^1/_3 (\sigma_\alpha + \sigma_\beta + \sigma_n)$$

The components $\sigma_a$, $\tau_{an}$, $\tau_{\beta n}$ must be continuous at the passage through $S$, that is

$$[\sigma_n] = [\tau_{\alpha n}] = [\tau_{\beta n}] = 0 \qquad \text{на } S \tag{2.2}$$

where a component in square brackets indicates the magnitude of its discontinuity at the passage through $S$.

Components $\sigma_a$, $\sigma_\beta$, $\tau_{a\beta}$ may be discontinuous.

Denoting by $n_1 = \cos\phi_1$, $n_2 = \cos\phi_2$, $n_3 = \cos\theta$, we write relationships (2.2) in the form

$$\left.\begin{aligned} \sigma^+ + \cos^2\theta^+ &= \sigma^- + \cos^2\theta^- \\ \cos\varphi_1{}^+ \cos\theta^+ &= \cos\varphi_1{}^- \cos\theta^- \\ \cos\varphi_2{}^+ \cos\theta^+ &= \cos\varphi_2{}^- \cos\theta^- \end{aligned}\right. \qquad \text{on } S \tag{2.3}$$

Since $\text{cas}^2\phi_1 + \cos^2\phi_2 = \sin^2\theta$, the latter two of the relationships (2.3), if squared and added, will yield

$$\sin^2 2\theta^+ = \sin^2 2\theta^- \tag{2.4}$$

It is easily shown that the following relationships must be satisfied between the angles, if (2.4) holds

$$\theta^+ = \begin{cases} \pm\,\theta^- \pm \pi m + {}^1/_2\pi \\ \pm\,\theta^- \pm \pi m \end{cases} \qquad (m = 0, 1, \ldots) \tag{2.5}$$

Relationships (2.5) lead to the equality

$$\cos\theta^+ = \pm \sin\,\theta^- \text{ for } \theta^+ = \pm\,\theta^- \pm \pi m + \tfrac{1}{2}\pi \tag{2.6}$$

and to the equality

$$\cos\theta^+ = \pm\cos\theta^- \tag{2.7}$$

for all the remaining cases. Note that in relationships (2.5), (2.6), and (2.7) the signs do not correspond. These signs are easily obtained for each concrete case.

It is easy to show that in case (2.7) the components $\sigma_\alpha$, $\sigma_\beta$, $\tau_{\alpha\beta}$ are continuous, and a consideration of only case (2.6) will be to the point. Then

$$\cos 2\theta^+ = -\cos 2\theta^- \tag{2.8}$$

From (2.3) and (2.6) we obtain

$$[\sigma] = \cos\, 2\theta^-, \qquad \begin{aligned} \cos\varphi_1{}^+ &= \pm\cos\varphi_1{}^-\operatorname{ctg}\theta^- \\ \cos\varphi_2{}^+ &= \pm\cos\varphi_2{}^-\operatorname{ctg}\theta^- \end{aligned} \tag{2.9}$$

It is easy to obtain

$$\begin{aligned} [\sigma_\alpha] &= \cos 2\theta^-\left(1 + \frac{\cos^2\varphi_1{}^-}{\sin^2\theta^-}\right), \\ [\sigma_\beta] &= \cos 2\theta^-\left(1 + \frac{\cos^2\varphi_2{}^-}{\sin^2\theta^-}\right), \end{aligned} \qquad [\tau_{\alpha\beta}] = \cos 2\theta^- \frac{\cos\varphi_1{}^-\cos\varphi_2{}^-}{\sin^2\theta^-} \tag{2.10}$$

Let us show that if the plastic state of stress on either side of the surface $S$ corresponds to one and the same edge of the Tresca-Saint Venant prism, only the second case should be considered in relations (2.6).

$$[\theta] = -2\theta^- \pm \pi m + \frac{\pi}{2} \qquad (m = 0, 1, \ldots)$$

In fact, it follows from (2.10) that for $\theta^- = 1/4\pi$

$$[\sigma_\alpha] = [\sigma_\beta] = [\tau_{\alpha\beta}] = 0$$

which indicates that there exists a continuous distribution of stresses; and, consequently, the third principal direction does not change its original orientation at the passage through the surface $S$. This is in conflict with the first relationship of (2.6)

$$[\theta] = \frac{\pi}{2} \pm \pi m \qquad (m = 0, 1, \ldots)$$

In the case when the surface $S$ separates regions of plastic state of stress which correspond to different edges of the Tresca-Saint Venant prism, then on the contrary, in view of analogous reasoning, let us consider only the first case of relationships (2.6), while the second case will be omitted.

Thence, the conditions on the surfaces of discontinuity $S$ may be written in final form as

$$[\sigma] = \cos 2\theta^-, \qquad [\theta] = -2\theta^- \pm \pi m + \frac{\pi}{2} \tag{2.11}$$

when the plastic state of stress on each side of the surface $S$ corresponds to one and the same edge of the Tresca-Saint Venant prism, and

$$[\sigma] = \cos 2\theta^{-},$$

$$[\theta] = \tfrac{1}{2}\pi \pm \pi m \qquad (2.12)$$

for the case when the plastic stress states (which are separated by surface $S$) correspond to different edges of Tresca-Saint Venant prism.

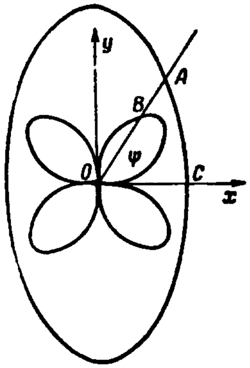


Fig. 1.

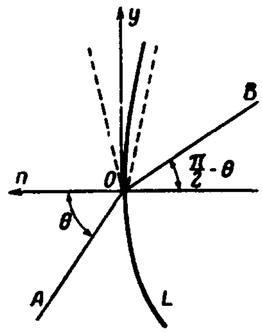


Fig. 2.

If axes $\alpha$ and $\beta$ are so directed, that $\cos \phi_1^- = 0$, then $\cos^2\phi_2 = \sin^2\theta^-$. For such a case denote axes $\alpha$ and $\beta$ by $x$ and $y$. Then, from relationship (2.10), it follows that

$$[\sigma_x] = a, \quad [\sigma_y] = 2a, \quad [\tau_{xy}] = 0 \qquad (a = \cos 2\theta^-) \qquad (2.13)$$

Let $\psi$ be the angle between axes $\alpha$ and $x$. We have then

$$\sigma_\alpha = \sigma_x \cos^2\psi + \sigma_y \sin^2\psi + \tau_{xy} \sin 2\psi$$

$$\tau_{\alpha\beta} = {}^1/_2 (\sigma_y - \sigma_x) \sin 2\psi + \tau_{xy} \cos 2\psi \qquad (2.14)$$

From (2.13) and (2.14) it easily follows

$$[\sigma_\alpha] = a(1 + \sin^2\psi), \quad [\sigma_\beta] = a(1 + \cos^2\psi), \quad [\tau_{\alpha\beta}] = a \sin\psi \cos\psi \qquad (2.15)$$

The change of magnitudes of discontinuities for components $\sigma_2$ and $\tau_{\alpha\beta}$ in the tangential plane to the surface $S$ is shown in Fig. 1. Point $A$ traces the curve of change of discontinuity of the component $\sigma_2$, and the Point $B$ indicates the same for the component $\tau_{\alpha\beta}$.

Note that from (2.15) it follows that

$$([\sigma_\alpha] - [\sigma_\beta])^2 + 4[\tau_{\alpha\beta}]^2 = a^2$$

Now consider the change in orientation of the third principal direction at the passage through the surface $S$. From the second relationship of (2.9) it is clear that $\cos \phi_1^+ = 0$ whenever $\cos \phi_1^- = 0$. This indicates that the direction of the principal stress $\sigma_3$ is always situated in the *noy* plane.

The change of direction of the third principal stress at the passage through the $S$ surface is shown in Fig.2. The direction of the third principal stress is indicated by rays $OA$ and $OB$. The line $L$ is the intersection line of the $noy$ plane and the surface $S$. If following Prager, the first and the second directions of shearing are introduced in the $noy$ plane, it is then easily observed that the direction of the discontinuity line $L$, in each of its points, bisects the angle formed by the intersection at that point of slip lines of the first family. These lines are indicated by dotted lines in Fig.2.

Consider the second case. Assume that

$$\sigma_1^+ = \sigma_2^+ = \sigma_3^+ - 1, \qquad \sigma_1^- = \sigma_2^- = \sigma_3^- + 1 \tag{2.16}$$

Then

$$\sigma_\alpha^+ = \sigma_1^+ + \cos^2\varphi_1^+, \qquad \tau_{\alpha\beta}^+ = \cos\varphi_1^+\cos\varphi_2 \ , \ldots$$
$$\sigma_\alpha^- = \sigma_1^- - \cos^2\varphi_1^-, \qquad \tau_{\alpha\beta}^- = -\cos\varphi_1^-\cos\varphi_2^-, \ldots$$
$$\sigma_1^+ = \sigma^+ - \frac{1}{3}, \qquad \sigma_1^- = \sigma^- + \frac{1}{3}$$

From the continuity condition for the components $\sigma_n$, $\tau_{\alpha n}$, $\tau_{\beta n}$ we obtain

$$\begin{aligned} \sigma^+ + \cos^2\theta^+ &= \sigma^- - \cos^2\theta^- + \frac{2}{3} \\ \cos\varphi_1^+\cos\theta^- &= -\cos\varphi_1^-\cos\theta^- \\ \cos\varphi_2^+\cos\theta^+ &= -\cos\psi_2^-\cos\theta^- \end{aligned} \tag{2.17}$$

Obviously, in this case too, the direction of the third principal stress is situated in the $noy$ plane.

Since $\sin 2\theta^+ = \sin 2\theta^-$, the relationships (2.5) to (2.8) are valid. Obviously, the discontinuities arise in all the cases.

For $\theta^+ = \pm\theta^- + \frac{1}{2}\pi + \pi m, \qquad \cos\theta^+ = \pm\sin\theta^-$

we will have

$$[\sigma] = -\frac{1}{3}, \qquad [\sigma_\alpha] = -1 + \frac{\cos^2\varphi_1^-}{\sin^2\theta^-}$$
$$[\tau_{\alpha\beta}] = \frac{\cos\varphi_1^-\cos\varphi_2^-}{\sin^2\theta^-}, \qquad [\sigma_\beta] = -1 + \frac{\cos^2\varphi_2^-}{\sin^2\theta^-} \tag{2.18}$$

From (2.18) it follows that

$$[\sigma_x] = -1, \qquad [\sigma_y] = 0, \qquad [\tau_{xy}] = 0$$
$$[\sigma_\alpha] = -\cos^2\psi, \qquad [\sigma_\beta] = -\sin^2\psi, \qquad [\tau_{\alpha\beta}] = \sin\psi\cos\psi$$

It is obvious that

$$([\sigma_\alpha] - [\sigma_\beta])^2 + 4[\tau_{\alpha\beta}]^2 = 1$$

A diagram of change of discontinuities, analogous to the one shown in Fig.1, can easily be pictured.

For $$\theta^+ = \pm\theta^- \pm \pi m, \qquad \cos\theta^+ = \pm\cos\theta^- \tag{2.19}$$

we have

$$[\sigma] = -2\cos^2\theta^- + \tfrac{2}{3}, \quad [\sigma_\alpha] = -2(\cos^2\theta^- - \cos^2\varphi_1^-)$$
$$[\tau_{\alpha\beta}] = 2\cos\varphi_1^-\cos\varphi_2^-, \quad [\sigma_\beta] = -2(\cos^2\theta^- - \cos^2\varphi_2^-) \tag{2.20}$$

From (2.20) it follows that

$$[\sigma_x] = -(1+a), \quad [\sigma_y] = -2a, \qquad [\tau_{xy}] = 0$$
$$[\sigma_\alpha] = -2a - (1-a)\cos^2\psi, \qquad [\sigma_\beta] = -2a - (1-a)\sin^2\psi$$
$$[\tau_{\alpha\beta}] = (1-a)\sin\psi\cos\psi$$
$$([\sigma_\alpha] - [\sigma_\beta])^2 + 4[\tau_{\alpha\beta}]^2 = (1-a)^2$$

In this case also it is easy to imagine a picture of variation of discontinuities, analogous to one shown in Fig. 1.

Note that for the case (2.19), (2.20) the direction of the third principal stress either does not change at the passage through the surface $S$, or the direction of the discontinuity line in the *noy* plane is the bisectrix of an angle which is formed by the directions of the third principal stress.

Now we pass to some examples. Planes of discontinuity which separate regions of constant stresses are of greatest applicability for the construction of discontinuous solutions for space problems. Let us develop a discontinuous state of stress for a four-sided pyramid whose cross-section perpendicular to its height is a rhombus (Fig. 3).

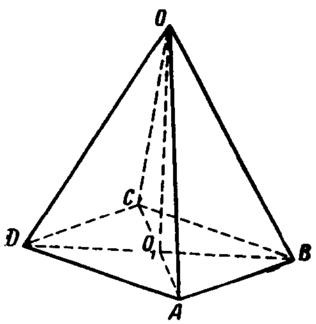


Fig. 3.

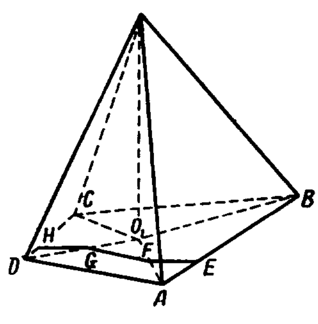


Fig. 4.

Assume that normal stresses act on the faces of this pyramid. Introduce planes of discontinuity $AOO_1$, $BOO_1$, $COO_1$, $DOO_1$. Apply relationships (2.19) and (2.20). Identify by index (1) components in the Region $ABOO_1$; use index (2) for region $BCOO_1$ and so on. Assume that within the region $ABOO_1$, the positive sign is used for $2k$ in relationship (2.1); within the region $BCOO_1$ the negative sign will appear and so on. Hence obtain

$$\sigma_{(1)} = \sigma_{(2)} - 2\cos^2\theta_{(1)} + \tfrac{2}{3}, \quad \sigma_{(3)} = \sigma_{(4)} - 2\cos^2\theta_{(3)} + \tfrac{2}{3}$$
$$\sigma_{(2)} = \sigma_{(3)} + 2\cos^2\theta_{(2)} - \tfrac{2}{3}, \quad \sigma_{(4)} = \sigma_{(1)} + 2\cos^2\theta_{(4)} - \tfrac{2}{3} \qquad (2.21)$$

Adding relationships (2.21) and considering that $\theta_{(1)} = \theta_{(2)}$, $\theta_{(3)} = \theta_{(4)}$, obtain

$$\cos^2\theta_{(1)} = \cos^2\theta_{(2)}$$

from which it follows that $\theta_1 = \theta_2$ and, consequently, the rhombus $ABCD$ must be a square. If $2\gamma$ denoted the angle between the faces $ABO$ and $CDO$, then, as is easily seen,

$$\cos\theta = \frac{\sqrt{2}}{2}\cos\gamma \qquad (2.22)$$

where the subscript for the quantity $\theta$ is omitted.

Assume that the faces $BCO$ and $DAO$ are stress free. Then $\sigma_3 = 0$ in the regions adjacent to these faces. Using relationships (2.16), (2.20), (2.22) obtain the unknown value of the constant normal pressure which acts on faces $ABO$ and $CDO$ :

$$p = -2k\,(2 - \cos^2\gamma)$$

In an entirely analogous way a pyramid with an arbitrary number of faces may be investigated. Following [5], it can be shown that the number of intersecting discontinuity planes at a single straight line must be not less than four.

Consider an example which would generalize the familiar presentation of the discontinuous solution for a truncated wedge [5]. Imagine a four-sided pyramid whose cross-section perpendicular to the height is a regular traperium, Fig.4. By increasing the height $OO_1$ and leaving the contour $ABCD$ unchanged, a prism is obtained which was considered in [5] under the conditions of plane deformation. Introduce the discontinuity surfaces $AOO_1$, $BOO_1$, $COO_1$, $DOO_1$. Assume that the direction of the third principal stress in the region $ABOO_1$ is perpendicular to the face $AOB$. Direction of the third principal stress in region $ADOO_1$ will be determined by the conditions at the discontinuity plane. If the face $ADO$ is parallel to the direction of the third principal stress, then the shear stresses in its plane will be equal to zero. Further, if passing through the discontinuity plane $DOO_1$, the direction of the third principal stress is parallel to the face $DCO$, in such a way that its projection on the $ABCD$ plane is perpendicular to the line $CB$, then only normal stresses will act on the face $DCO$. Inasmuch as the stresses in the region $ABOO_1$ are related in an elementary manner to the stresses in the adjacent regions, it follows, assuming that the stresses acting on the face $ADO$ are equal to zero, that the normal stresses on faces $ABC$ and

*DCO* may be determined.

The development indicated is based on an elementary idea and is made possible through application of conditions (2.11). It leads to cumbersome transcendental relationships between the angles, which are characteristic of the pyramid, and are omitted here.

The projection of the direction of the third principal stress on the *ABCD* plane is indicated by line *EFGH* in Fig. 4.

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